An Introduction to Lie Groups for State Estimation

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Context

- In robotics, **rotations** and **poses** are everywhere in the theory
- Localization, SLAM, Calibration, ...

Primer on State Estimation

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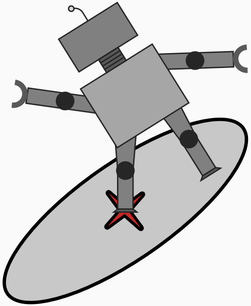
State Estimation Principles



State estimation consists in inferring a “state” $\boldsymbol{x}$ from

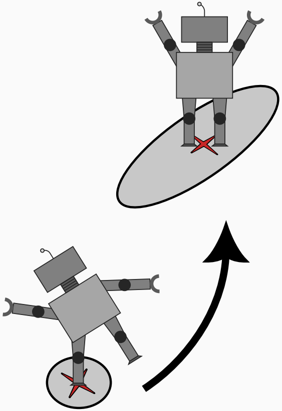
1. Some measurements $\boldsymbol{y}$
2. A prior on the system dynamics $\boldsymbol{v}$

We represent uncertainty with a probability distribution $p(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{v})$.

In the Gaussian case:

$$\boldsymbol{x} \sim \mathcal{N}(\hat{\boldsymbol{x}}, \mathbf{P})$$

Prior



We assume the robot follow a linear motion model:

$$\boldsymbol{x}_k = \underbrace{\mathbf{A}_{k-1}\boldsymbol{x}_{k-1}}_{\text{Dynamics}} + \underbrace{\boldsymbol{v}_k}_{\text{Input}} + \underbrace{\boldsymbol{w}_k}_{\text{Noise}}, \quad \boldsymbol{w}_k \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q}_k)$$

It can be put in a batch form

$$\boldsymbol{x} = \mathrm{A}(\boldsymbol{v} + \boldsymbol{w})$$

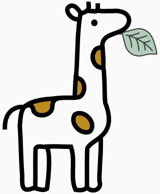
where $\boldsymbol{x}$ contains all the states $\boldsymbol{x}_k$ staked together, as

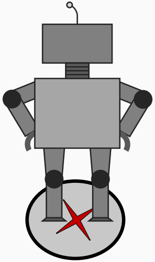
$$\boldsymbol{x} = \left[\boldsymbol{x}_0\ \boldsymbol{x}_1 \cdots \boldsymbol{x}_N\right]^T$$

The prior have the distrubution $p(\boldsymbol{x}|\boldsymbol{v}) = \mathcal{N}(\check{\mathbf{x}}, \check{\mathbf{P}})$ with

$$\check{\boldsymbol{x}} = \mathrm{A}\boldsymbol{v}$$
$$\check{\mathbf{P}} = \mathrm{AQA}^T$$

Measurements





We assume a linear measurement model:

$$\boldsymbol{y}_k = \mathbf{C}_k \boldsymbol{x}_k + \boldsymbol{n}_k \quad \boldsymbol{n}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$$

It can the put in a batch form:

$$\boldsymbol{y} = \mathbf{C}\boldsymbol{x} + \boldsymbol{n}$$

where $\boldsymbol{y}$ contains all the measurements $\boldsymbol{y}_k$ staked together, as

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_0 \; \boldsymbol{y}_1 \cdots \boldsymbol{y}_N \end{bmatrix}^T$$

What about the full posterior $p(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{v})$?

Bayesian Inference

With a bit of maths, we get the full posterior as

$$p(\boldsymbol{x}|\boldsymbol{v},\boldsymbol{y}) = \mathcal{N}\left(\underbrace{\left(\check{\mathbf{P}}^{-1} + \mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\right)^{-1}\left(\check{\mathbf{P}}^{-1}\check{\boldsymbol{x}} + \mathbf{C}^T\mathbf{R}^{-1}\boldsymbol{y}\right)}_{\text{Mean } \hat{\boldsymbol{x}}}, \underbrace{\left(\check{\mathbf{P}}^{-1} + \mathbf{C}^T\mathbf{R}^{-1}\mathbf{C}\right)^{-1}}_{\text{Covariance } \hat{\boldsymbol{P}}}\right) \tag{1}$$

For the mean, a more compact form is

$$\mathbf{H}^T\mathbf{W}^{-1}\mathbf{H}\,\hat{\boldsymbol{x}} = \mathbf{H}^T\mathbf{W}^{-1}\boldsymbol{z}$$

with

$$\boldsymbol{z} = \begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{y}\end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix}\mathbf{A}^{-1}\\ \mathbf{C}\end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix}\mathbf{Q} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{bmatrix}$$

RTS smoother

we can manipulate the matrices to get to a better form for solving efficiently

Forward pass (k=1...N)

$$
\begin{aligned}
\check{\mathbf{P}}_{k,f} &= \mathbf{A}_{k-1}\hat{\mathbf{P}}_{k-1,f}\mathbf{A}_{k-1}^T + \mathbf{Q}_k \\
\check{\boldsymbol{x}}_{k,f} &= \mathbf{A}_{k-1}\hat{\boldsymbol{x}}_{k-1,f} + \boldsymbol{v}_k \\
\mathbf{K}_k &= \check{\mathbf{P}}_{k,f}\mathbf{C}_k^T\left(\mathbf{C}_k\check{\mathbf{P}}_{k,f}\mathbf{C}_k^T + \mathbf{R}_k\right)^{-1} \\
\hat{\mathbf{P}}_{k,f} &= (\mathbf{I} - \mathbf{K}_k\mathbf{C}_k)\check{\mathbf{P}}_{k,f} \\
\hat{\boldsymbol{x}}_{k,f} &= \check{\boldsymbol{x}}_{k,f} + \mathbf{K}_k(\boldsymbol{y}_k - \mathbf{C}_k\check{\boldsymbol{x}}_{k,f})
\end{aligned}
$$

Kalman Filter

Backward pass (k=N...1)

$$
\begin{aligned}
\hat{\boldsymbol{x}}_{k-1} &= \hat{\boldsymbol{x}}_{k-1,f} + \left(\hat{\mathbf{P}}_{k-1,f}\mathbf{A}_{k-1}^T\check{\mathbf{P}}_{k,f}^{-1}\right)(\hat{\boldsymbol{x}}_k - \check{\boldsymbol{x}}_{k,f}) \\
\hat{\mathbf{P}}_{k-1} &= \hat{\mathbf{P}}_{k-1,f} + \left(\hat{\mathbf{P}}_{k-1,f}\mathbf{A}_{k-1}^T\check{\mathbf{P}}_{k,f}^{-1}\right)\left(\hat{\mathbf{P}}_k - \check{\mathbf{P}}_{k,f}\right) \\
&\quad \times \left(\hat{\mathbf{P}}_{k-1,f}\mathbf{A}_{k-1}^T\check{\mathbf{P}}_{k,f}^{-1}\right)^T
\end{aligned}
$$

The Rotation Issue

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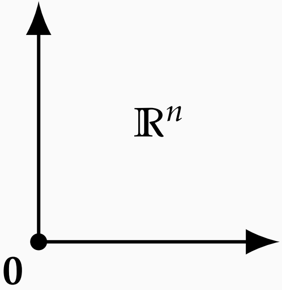
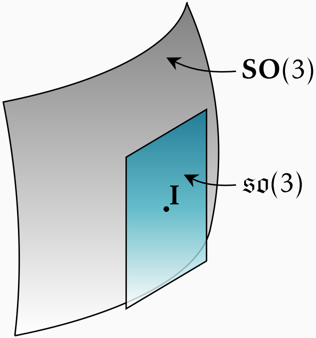
Dealing with rotations and poses



- Find a parametrization $p \in \mathbb{R}^n$ of the rotation and the pose
- A parametrization either brings constraints or singularities

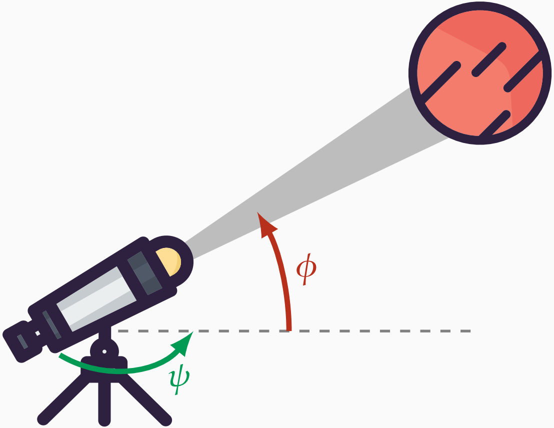


- Directly work with the full objects $\boldsymbol{C}$, $\boldsymbol{T}$
- Classier, more intuitive, but more complex...

A quick example in 2D: Gimbal Lock

Imagine a telescope tracking the trajectory of a planet. At the precise moment the planet passes directly overhead, the azimuth angle ψ must instantaneously jump by 180° to continue following the planet's motion.

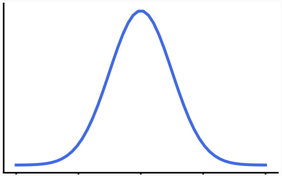
When $\phi = 180^\circ$, changes in the azimuth angle ψ no longer affect the state of the telescope. This results in a local loss of one degree of freedom.



The nice properties of the real numbers

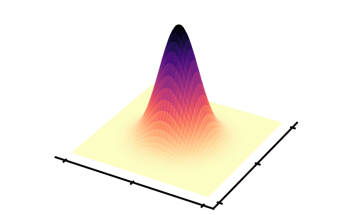
The real numbers $\mathbb{R}$

- Field with operations $(+, \times)$
- lots of nice properties: commutative, associative, ...



The N-dimensional space $\mathbb{R}^N$

- Vector space over $\mathbb{R}$
- lots of nice properties: commutative, associative, ...



Rotation matrices

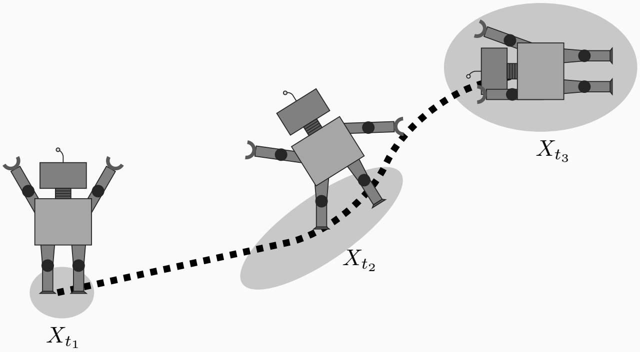
- Continuous Group
- **noncommutative**

?

Using probability distributions on SO(3)

- $\boldsymbol{C}_1 + \boldsymbol{C}_2$ is not a rotation matrix
- $\boldsymbol{C} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ does not make sense

We cannot use standard mathematical tools since **the rotations and poses do not form vector spaces.**



Lie Groups and Algebras

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Group

A group is a non-empty set G together with a binary operation on G, denoted "$\cdot$", such that the following four requirements are satisfied:

1. **Closure:** For all a, b in G, we have
$$a \cdot b \in G$$
2. **Identity element:** There exists an unique element e in G (identity element) such that, for every a in G, one has
$$e \cdot a = a \quad \text{and} \quad a \cdot e = a.$$
3. **Associativity:** For all a, b, c in G, one has
$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$
4. **Inverse element:** For each a in G, there exists an element b in G such that
$$a \cdot b = e \quad \text{and} \quad b \cdot a = e,$$
where e is the identity element. b is commonly denoted a^{-1}.

The 3D rotation group $\mathbf{SO}(3)$

The Special Orthogonal group $\mathbf{SO}(3)$ represents the rotations in 3D:

$\mathbf{SO}(3)$ group

$$\mathbf{SO}(3) = \left\{ \boldsymbol{C} \in \mathbb{R}^{3\times 3} \middle| \boldsymbol{C}\boldsymbol{C}^T = \boldsymbol{I}, \det \boldsymbol{C} = 1 \right\}$$

with the standard matrix multiplication as the group operation.

Prove that $\mathbf{SO}(3)$ follows the group axioms and thus is a group.

The 3D pose group $\mathbf{SE}(3)$

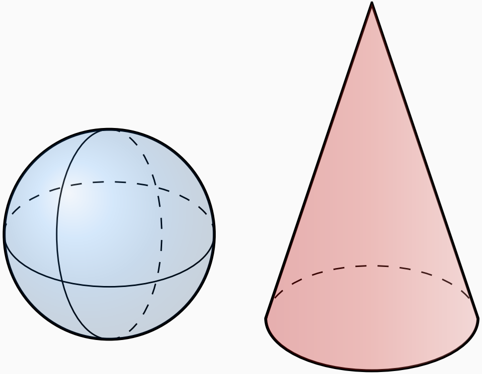
The Special Euclidean group $\mathbf{SE}(3)$ represent the spatial transformations in 3D:

$\mathbf{SE}(3)$ group

$$\mathbf{SE}(3) = \left\{ \begin{bmatrix} \boldsymbol{C} & \boldsymbol{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \;\middle|\; \boldsymbol{C} \in \mathbf{SO}(3), \boldsymbol{r} \in \mathbb{R}^3 \right\}$$

with the standard matrix multiplication as the group operation.

Prove that $\mathbf{SE}(3)$ follows the group axioms and thus is a group.



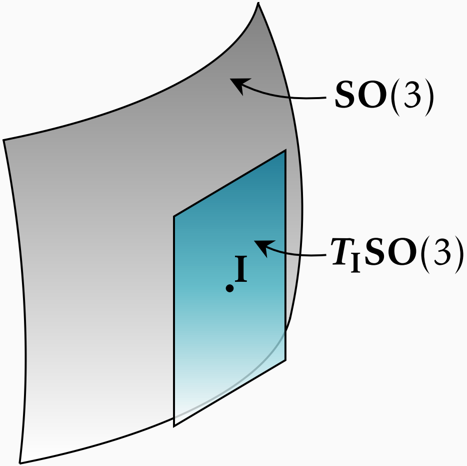
Lie Group

A Lie group is a group that is also a differentiable manifold.

Differentiable Manifold

A differentiable manifold is a type of manifold that is locally similar enough to a vector space to apply calculus.

Cartan's theorem if H is a closed subgroup of a Lie group G, then H is a Lie group



If $\mathbf{SO}(3)$ and $\mathbf{SE}(3)$ are Lie groups, we can locally approximate them with vector spaces...

Lie Algebra

A Lie Algebra is a vector space V with a binary operation $[\cdot, \cdot]$, called the **Lie bracket**, that satisfies:

1. **Closure:**

$$[X, Y] \in V, \text{ for all } X, Y \in V.$$

2. **Bilinearity:**

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$

$$[Z, aX + bY] = a[Z, X] + b[Z, Y],$$

where $a, b \in \mathbb{F}$ (the underlying field), and $X, Y, Z \in V$.

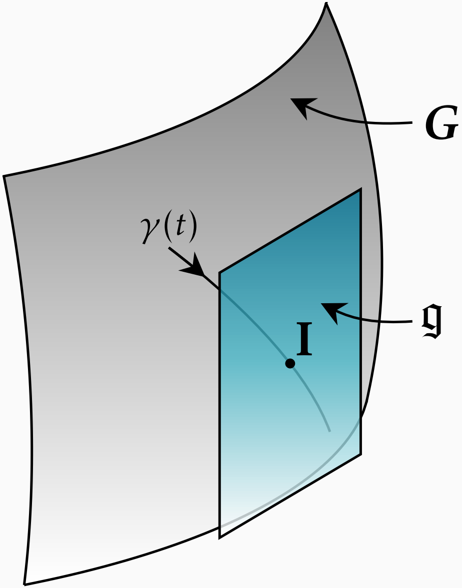
3. **Alternating:**

$$[X, X] = 0, \text{ for all } X \in V.$$

4. **Jacobi Identity:**

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in V$.



A Lie algebra $\mathfrak{g}$ of a Lie group G is the tangent space of G at the identity element (and equipped with the Lie bracket $[\cdot, \cdot]$).
The tangent space at the identity is defined as

$$\mathfrak{g} = \{\gamma'(0) \,|\, \gamma(t) \in G, t \in \mathbb{R}, \gamma(0) = \boldsymbol{I}\}$$

where $\gamma(\cdot)$ is a **one-parameter subgroup**.

Rotations

Lie algebra $\mathfrak{so}(3)$ associated with SO(3)

vectorspace: $\mathfrak{so}(3) = \{\Phi = \boldsymbol{\phi}^\wedge \in \mathbb{R}^{3\times 3} \mid \boldsymbol{\phi} \in \mathbb{R}^3\}$,

Lie bracket: $[\Phi_1, \Phi_2] = \Phi_1\Phi_2 - \Phi_2\Phi_1$, with

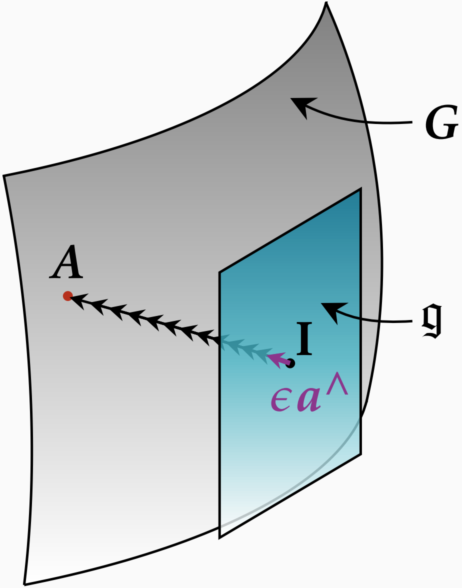
$$\boldsymbol{\phi}^\wedge = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}, \boldsymbol{\phi} \in \mathbb{R}^3.$$

Lie algebra $\mathfrak{se}(3)$ associated with $\mathrm{SE}(3)$

vectorspace: $se(3) = \{\Xi = \boldsymbol{\xi}^\wedge \in \mathbb{R}^{4\times4} \mid \boldsymbol{\xi} \in \mathbb{R}^6\}$,

Lie bracket: $[\Xi_1, \Xi_2] = \Xi_1\Xi_2 - \Xi_2\Xi_1$, with

$$\boldsymbol{\xi}^\wedge = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{bmatrix}^\wedge = \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}, \boldsymbol{\rho}, \boldsymbol{\phi} \in \mathbb{R}^3.$$



Close enough to the identity, we have the linear approximation

$$\boldsymbol{I} + \epsilon \boldsymbol{a}^{\wedge} \in \boldsymbol{G}, \quad \boldsymbol{a}^{\wedge} \in \mathfrak{g}$$

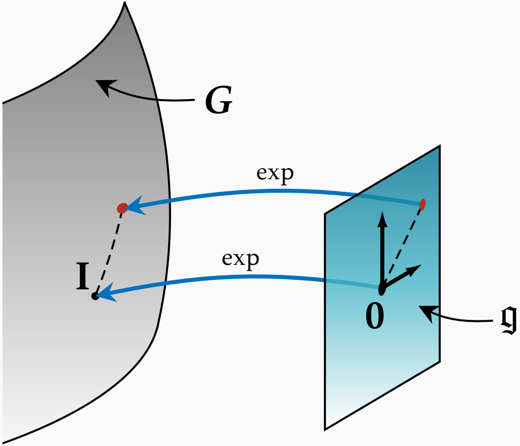
with ϵ being a very small number.

If we apply a large number of time this infinitesimal action, we can reach other elements of the group:

$$\begin{aligned} \boldsymbol{A} &= \lim_{n\to\infty} \left(\boldsymbol{I} + \frac{1}{n}\boldsymbol{a}^{\wedge} \right)^{n} \\ &= \exp\left(\boldsymbol{a}^{\wedge}\right) \end{aligned}$$

Exponential Map

The key property of a Lie algebra is that it **completely describes the structure** of its associated Lie Group. In the case of $\mathbf{SO}(3)$ and $\mathbf{SE}(3)$, we can find a local bijection between the Lie Group and the Lie Algebra, called the **exponential map**.



$$\exp(\mathbf{X}) = 1 + \mathbf{X} + \frac{1}{2!}\mathbf{X}^2 + \frac{1}{3!}\mathbf{X}^3 + \cdots = \sum_{n=0}^{\infty} \frac{\mathbf{X}^n}{n!},$$

$$\ln(\mathbf{A}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(\mathbf{A} - \mathbf{I})^n$$

Probability and Statistics

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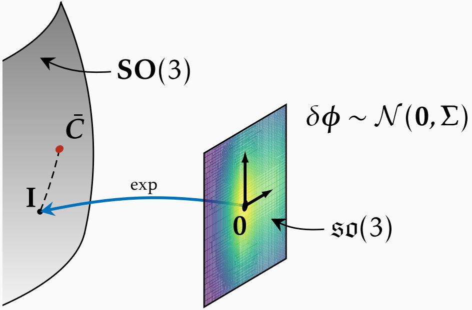
Probabilistic Rotation

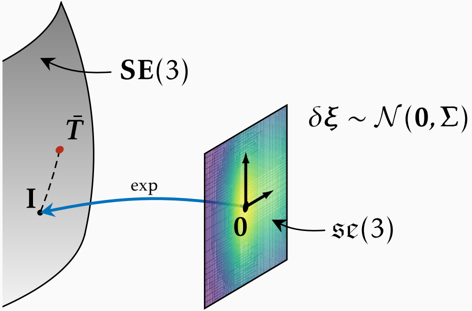


We want to model an uncertainty on a rotation $\boldsymbol{C} \in \mathbf{SO}(3)$:

$$\boldsymbol{C} = \exp\left(\delta\boldsymbol{\phi}^\wedge\right)\bar{\boldsymbol{C}}$$

with $\delta\boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$

Probabilistic Poses

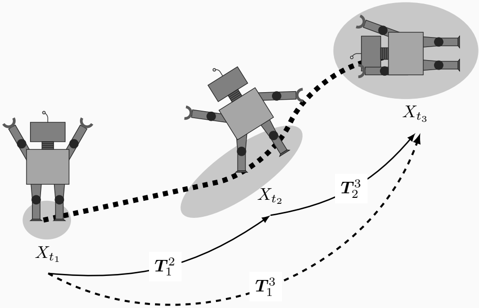


We want to model an uncertainty on a pose $\boldsymbol{T} \in \mathbf{SE}(3)$:

$$\boldsymbol{T} = \exp\left(\delta\boldsymbol{\xi}^{\wedge}\right) \bar{\boldsymbol{T}}$$

with $\delta\boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$

Compounding Poses



We want to model an uncertainty on a compounded pose $\boldsymbol{T} \in \mathbf{SE}(3)$: Given

$$\boldsymbol{T}_1^2 = \exp\left(\delta\boldsymbol{\xi}_1^\wedge\right)\bar{\boldsymbol{T}}_1^2, \quad \boldsymbol{\xi}_1 \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_1)$$
$$\boldsymbol{T}_2^3 = \exp\left(\delta\boldsymbol{\xi}_2^\wedge\right)\bar{\boldsymbol{T}}_2^3, \quad \boldsymbol{\xi}_2 \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_2)$$

find the mean and covariance of the compounded pose

$$\begin{aligned} \boldsymbol{T}_1^3 &= \boldsymbol{T}_1^2\boldsymbol{T}_2^3 \\ &= \exp\left(\delta\boldsymbol{\xi}_1^\wedge\right)\bar{\boldsymbol{T}}_1^2\exp\left(\delta\boldsymbol{\xi}_2^\wedge\right)\bar{\boldsymbol{T}}_2^3 \\ &\approx \exp\left(\Big[\underbrace{\delta\boldsymbol{\xi}_1 + \boldsymbol{\mathcal{T}}_1^2\delta\boldsymbol{\xi}_2}_{\sim\mathcal{N}(\mathbf{0},\Sigma_1+\boldsymbol{\mathcal{T}}_1^2\Sigma_2\boldsymbol{\mathcal{T}}_1^{2\,T})}\Big]^\wedge\right)\underbrace{\bar{\boldsymbol{T}}_1^2\bar{\boldsymbol{T}}_2^3}_{\text{Mean}} \end{aligned}$$

Kinematics

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Suppose we have a perturbed rotation matrix $\boldsymbol{C}'$ such that

$$\boldsymbol{C}' = \exp\left(\delta\boldsymbol{\phi}^\wedge\right)\boldsymbol{C}$$

Assuming the perturbation is small, we simplify this into

$$\begin{aligned}
&\boldsymbol{C}' \approx (\boldsymbol{I} + \delta\boldsymbol{\phi}^\wedge)\boldsymbol{C} \\
\Rightarrow &\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{C}' \approx \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{I} + \delta\boldsymbol{\phi}^\wedge\right)\boldsymbol{C} \\
\Leftrightarrow &(\boldsymbol{\omega} + \delta\boldsymbol{\omega})^\wedge(\boldsymbol{I} + \delta\boldsymbol{\phi}^\wedge)\boldsymbol{C} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{I} + \delta\boldsymbol{\phi}^\wedge\right)\boldsymbol{C} \\
\Rightarrow &\delta\dot{\boldsymbol{\phi}} = \boldsymbol{\omega}^\wedge\delta\boldsymbol{\phi} + \delta\boldsymbol{\omega}
\end{aligned}$$

Linearized kinematics on SO(3)

$$\begin{cases} \text{Nominal:} & \dot{\boldsymbol{C}} = \boldsymbol{\omega}^\wedge\boldsymbol{C} \\ \text{Perturbation:} & \delta\dot{\boldsymbol{\phi}} = \boldsymbol{\omega}^\wedge\delta\boldsymbol{\phi} + \delta\boldsymbol{\omega} \end{cases}$$

Linearized kinematics on SE(3)

$$\begin{cases} \text{Nominal:} & \dot{\boldsymbol{T}} = \boldsymbol{\varpi}^\wedge\boldsymbol{T} \\ \text{Perturbation:} & \delta\dot{\boldsymbol{\xi}} = \boldsymbol{\varpi}^\curlywedge\delta\boldsymbol{\xi} + \delta\boldsymbol{\varpi} \end{cases}$$

Filtering on Lie Groups

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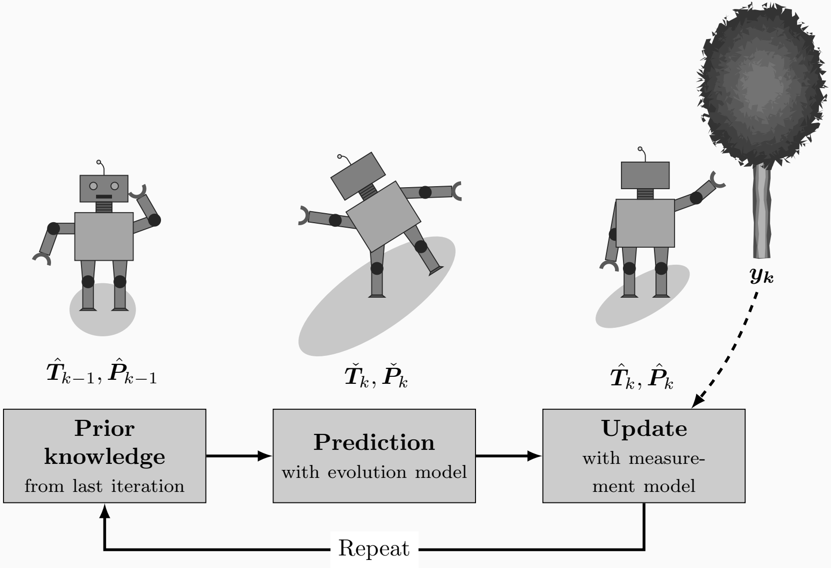
Kalman Filter on Lie Groups



Kalman Filter - Prediction Step

Let $\boldsymbol{C}' = \exp\left(\delta\boldsymbol{\phi}^\wedge\right)\boldsymbol{C}, \quad \delta\boldsymbol{\phi} \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{P}\right)$

1) Use the nominal equation to propagate the mean

$$\dot{\boldsymbol{C}} = \boldsymbol{\omega}^\wedge \boldsymbol{C}$$
$$\Rightarrow \check{\boldsymbol{C}}_k = \underbrace{\exp\left(\Delta t \cdot \boldsymbol{\omega}_k^\wedge\right)}_{\boldsymbol{F}_k} \boldsymbol{C}_{k-1}$$

2) Use the perturbation equation to propagate the co-variance

$$\delta\dot{\boldsymbol{\phi}} = \boldsymbol{\omega}^\wedge \delta\boldsymbol{\phi} + \delta\boldsymbol{\omega}$$
$$\Rightarrow \delta\boldsymbol{\phi}_k = \underbrace{\exp\left(\Delta t \cdot \boldsymbol{\omega}_k^\wedge\right)}_{\boldsymbol{F}_k} \delta\boldsymbol{\phi}_{k-1} + \boldsymbol{w}_k$$

Thus

$$\check{\boldsymbol{P}}_k = \boldsymbol{F}_k \boldsymbol{P}_{k-1} \boldsymbol{F}_k^T + \boldsymbol{Q}_k$$

KF Filter - Prediction step

$$\begin{cases} \check{\boldsymbol{C}}_k &= \boldsymbol{F}_k \boldsymbol{C}_{k-1} \\ \check{\boldsymbol{P}}_k &= \boldsymbol{F}_k \boldsymbol{P}_{k-1} \boldsymbol{F}_k^T + \boldsymbol{Q}_k \end{cases}$$

Kalman Filter - Update Step

We write the measurement model as

$$\boldsymbol{y}_k = \boldsymbol{G}_k \delta\boldsymbol{\phi}_k + \boldsymbol{n}_k + \text{cst}, \quad \boldsymbol{n}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_k)$$

The Kalman gain and covariance update equations are then unchanged from the generic case:

$$\begin{aligned} \boldsymbol{K}_k &= \check{\boldsymbol{P}}_k \boldsymbol{G}_k^T \left(\boldsymbol{G}_k \check{\boldsymbol{P}}_k \boldsymbol{G}_k^T + \boldsymbol{R}_k \right)^{-1} \\ \boldsymbol{\mu}_{\delta\boldsymbol{\phi}_k} &= \boldsymbol{K}_k \left(\boldsymbol{y}_k - \check{\boldsymbol{y}}_k \right) \\ \boldsymbol{P}_{\delta\boldsymbol{\phi}_k} &= \left(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{G}_k \right) \check{\boldsymbol{P}}_k \end{aligned}$$

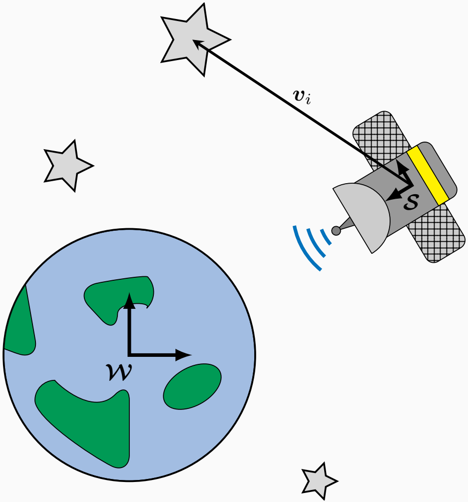
From this, we have the updated estimate

$$\begin{aligned} \hat{\boldsymbol{C}}_k &= \exp\left(\delta\hat{\boldsymbol{\phi}}_k + \boldsymbol{\mu}_{\delta\boldsymbol{\phi}_k} \right) \boldsymbol{C}_k, \quad \delta\hat{\boldsymbol{\phi}}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{P}_{\delta\boldsymbol{\phi}_k}) \\ &\approx \underbrace{\exp\left(\boldsymbol{J}(\boldsymbol{\mu}_{\delta\boldsymbol{\phi}_k}) \delta\hat{\boldsymbol{\phi}}_k \right)}_{\text{uncertainty}} \underbrace{\exp\left(\boldsymbol{\mu}_{\delta\boldsymbol{\phi}_k} \right) \boldsymbol{C}_k}_{\text{updated mean}} \end{aligned}$$

KF Filter - Update step

$$\begin{cases} \boldsymbol{K}_k &= \check{\boldsymbol{P}}_k \boldsymbol{G}_k^T \left(\boldsymbol{G}_k \check{\boldsymbol{P}}_k \boldsymbol{G}_k^T + \boldsymbol{R}_k \right)^{-1} \\ \boldsymbol{\mu}_{\delta\boldsymbol{\phi}_k} &= \boldsymbol{K}_k \left(\boldsymbol{y}_k - \check{\boldsymbol{y}}_k \right) \\ \boldsymbol{P}_{\delta\boldsymbol{\phi}_k} &= \left(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{G}_k \right) \check{\boldsymbol{P}}_k \\ \hat{\boldsymbol{C}}_k &= \exp\left(\boldsymbol{\mu}_{\delta\boldsymbol{\phi}_k}^{\wedge} \right) \check{\boldsymbol{C}}_k \\ \hat{\boldsymbol{P}}_k &= \boldsymbol{J}(\boldsymbol{\mu}_{\delta\boldsymbol{\phi}_k}) \boldsymbol{P}_{\delta\boldsymbol{\phi}_k} \boldsymbol{J}(\boldsymbol{\mu}_{\delta\boldsymbol{\phi}_k})^T \end{cases}$$

Example: Attitude Estimation



A satellite estimates its orientation $\boldsymbol{C} \in \mathbf{SO}(3)$.

We estimate the rotation via

$$\boldsymbol{C} = \exp(\delta\boldsymbol{\phi})\hat{\boldsymbol{C}}$$

with $\delta\boldsymbol{\phi} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{P})$.

It measures known star directions $\boldsymbol{v}_k$:

$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{v}_k + \boldsymbol{n}_k$$

Kalman Equations

Prior

The continuous model is

$$\dot{\boldsymbol{C}} = \boldsymbol{\omega}^{\wedge}\boldsymbol{C}$$

Integrating the ODE, we get

$$\check{\boldsymbol{C}}_k = \exp(\boldsymbol{\omega}_k \Delta t)\hat{\boldsymbol{C}}_{k-1}$$

and thus

$$\boldsymbol{F}_k = \exp(\boldsymbol{\omega}_k \Delta t)$$

Update

From the measurement model

$$\boldsymbol{y}_k = \boldsymbol{C}_k \boldsymbol{v}_k + \boldsymbol{n}_k$$

We linearize

$$\begin{aligned} \boldsymbol{y}_k &= (\boldsymbol{I} + \delta\boldsymbol{\phi}_k^{\wedge})\check{\boldsymbol{C}}_k \boldsymbol{v}_k + \boldsymbol{n}_k \\ \boldsymbol{y}_k &= \check{\boldsymbol{C}}_k \boldsymbol{v}_k - (\check{\boldsymbol{C}}_k \boldsymbol{v}_k)^{\wedge}\delta\boldsymbol{\phi} + \boldsymbol{n}_k \end{aligned}$$

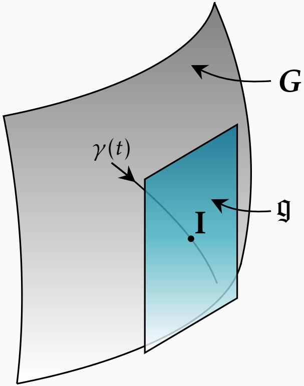
and thus

$$\boldsymbol{G}_k = -(\check{\boldsymbol{C}}_k \boldsymbol{v}_k)^{\wedge}$$

Summary

Primer on State Estimation
The Rotation Issue
Lie Groups and Algebras
Probability and Statistics
Kinematics
Filtering on Lie Groups
Summary

Summary



- Lie theory has numerous uses in robotics, such as control, state estimation, and modeling.
- Using this theory, we can define probability distributions on rotations and poses, as well as perform optimization.
- State Estimation is cleaner and “singularity-free”.

To dive further in the theory:

Barfoot, T. D. (2024). State estimation for robotics. Cambridge University Press.