Dynamic modelling of serial robots

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Lecture

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Outline

Why is dynamics so important in Robotics?

Dynamics with Lagrange equations

Concluding remarks

What is dynamics?

Definition

Dynamics is a branch of physics (specifically classical mechanics) concerned with the study of forces and torques and their effect on motion, as opposed to kinematics, which studies the motion of objects without reference to its causes.

Robotics and Dynamics, a few application examples

What things in common have these applications?

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- in the controller

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The use of the dynamic models

- in the controller
- during the design process (e.g. for choosing the actuators / lowering their torques)
- for simulating the robot motion (during the design phase, but also when preparing the process, etc.)

Aim of the present lecture

To study the dynamics of mechanisms, and more specifically here, of serial robot architectures

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- the inverse dynamic model: $\boldsymbol{\tau} = \mathbf{idm}(\mathbf{q}_a, \dot{\mathbf{q}}_a, \ddot{\mathbf{q}}_a, \mathbf{w}_e)$
- the direct dynamic model: $\ddot{\mathbf{q}}_a = \mathbf{ddm}(\dot{\mathbf{q}}_a, \mathbf{q}_a, \boldsymbol{\tau}, \mathbf{w}_e)$

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- the direct dynamic model: $\ddot{\mathbf{q}}_a = \mathbf{ddm}(\dot{\mathbf{q}}_a, \mathbf{q}_a, \boldsymbol{\tau}, \mathbf{w}_e)$

To be used in control, these models must be accurate

Can models be accurate?

Dynamic model of the Orthoglide

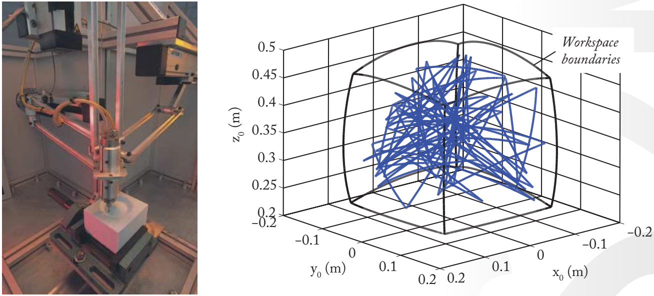
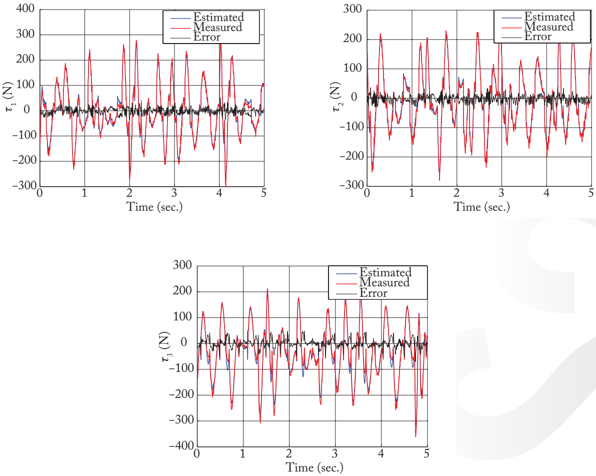


Figure: Orthoglide + trajectory.

Can models be accurate?

Dynamic model of the Orthoglide



The Lagrange formalism

The Lagrange equations

$$\boldsymbol{\tau} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T - \left(\frac{\partial L}{\partial \mathbf{q}}\right)^T \quad (1)$$

where

- $\boldsymbol{\tau}$ are the vector of generalized forces applied on the system,
- $\mathbf{q}$ is the vector of generalized coordinates. For rigid robots, $\mathbf{q} = \mathbf{q}_a$.
- $\dot{\mathbf{q}}$ is the vector of generalized velocities,
- L is a function called the Lagragian:

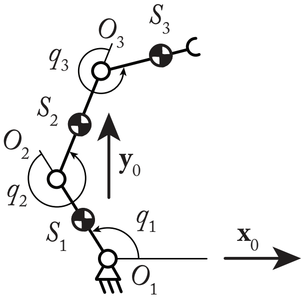
$$L = E - U \quad (2)$$

in which E is the kinetic energy of the system and U its potential energy (due to gravity effects, deformations, etc.).

The Lagrange formalism

An introductory example

Let us consider a planar 3R serial robot.

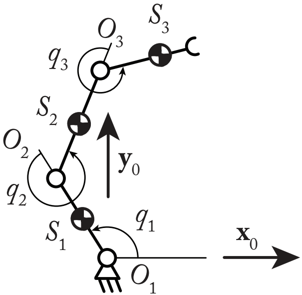


- S_i aligned along $\overrightarrow{O_i O_{i+1}}$ $\Rightarrow my_i = 0$ and $mz_i = 0$
- bodies are symmetrical $\Rightarrow xy_i = 0$, $xz_i = 0$ and $yz_i = 0$
- gravity is along $\mathbf{z}_0$ $\Rightarrow$ Potential energy is constant for any robot configuration
- $S_3 \equiv O_3 \Rightarrow mx_3 = 0$

The Lagrange formalism

An introductory example

Computation of the kinetic energy

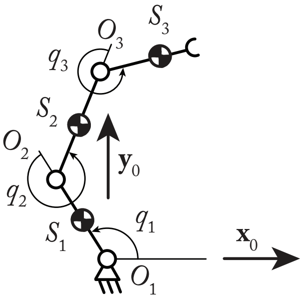


Recall

$$E_i = \frac{1}{2}\left(m_i\, {}^i\mathbf{v}_i^T\, {}^i\mathbf{v}_i + {}^i\boldsymbol{\omega}_i^T\, {}^i\mathbf{I}_{O_i}\, {}^i\boldsymbol{\omega}_i + 2\,{}^i\mathbf{ms}_i^T\left({}^i\mathbf{v}_i \times {}^i\boldsymbol{\omega}_i\right)\right) \quad (3)$$

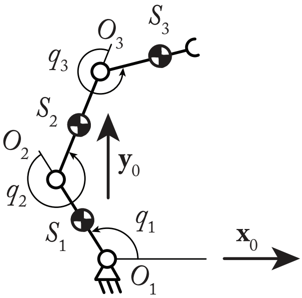
$$\text{with } {}^i\mathbf{ms}_i^T = \begin{bmatrix} mx_i & 0 & 0 \end{bmatrix} \quad (4)$$

The Lagrange formalism



- For body $\mathcal{B}_1$, $\mathbf{v}_1 = \mathbf{0}, \quad \boldsymbol{\omega}_1 = \dot{q}_1 \mathbf{z}_0$

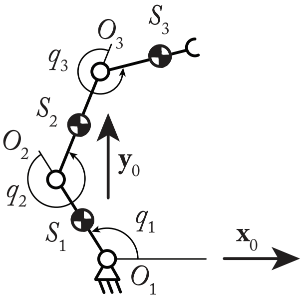
The Lagrange formalism



- For body $\mathcal{B}_1$, $\mathbf{v}_1 = \mathbf{0}, \quad \boldsymbol{\omega}_1 = \dot{q}_1 \mathbf{z}_0$
- For body $\mathcal{B}_2$, $\boldsymbol{\omega}_2 = (\dot{q}_1 + \dot{q}_2)\mathbf{z}_0$ and

$$\mathbf{v}_2 = \frac{d}{dt}\Big|_{\mathcal{F}_0}(\overrightarrow{O_1 O_2}) = \ell_{O_1 O_2} \mathbf{y}_1 \dot{q}_1$$

The Lagrange formalism



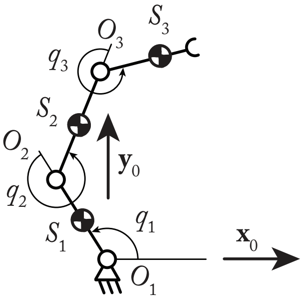
- For body $\mathcal{B}_1$, $\mathbf{v}_1 = \mathbf{0}$, $\boldsymbol{\omega}_1 = \dot{q}_1 \mathbf{z}_0$
- For body $\mathcal{B}_2$, $\boldsymbol{\omega}_2 = (\dot{q}_1 + \dot{q}_2)\mathbf{z}_0$ and

$$\mathbf{v}_2 = \frac{d}{dt}|_{\mathcal{F}_0}(\overrightarrow{O_1 O_2}) = \ell_{O_1 O_2} \mathbf{y}_1 \dot{q}_1$$

- For body $\mathcal{B}_3$, $\boldsymbol{\omega}_3 = (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)\mathbf{z}_0$

$$\begin{aligned}\mathbf{v}_3 &= \frac{d}{dt}|_{\mathcal{F}_0}(\overrightarrow{O_0 O_2}) \\ &= \frac{d}{dt}|_{\mathcal{F}_0}(\overrightarrow{O_1 O_2} + \overrightarrow{O_2 O_3}) \\ &= \ell_{O_1 O_2} \mathbf{y}_1 \dot{q}_1 + \ell_{O_2 O_3} \mathbf{y}_2 (\dot{q}_1 + \dot{q}_2)\end{aligned}$$

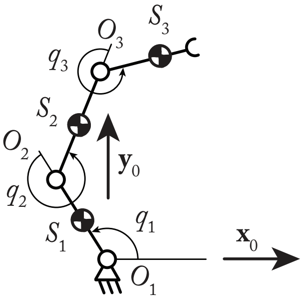
The Lagrange formalism



Energy computation:

- For body $\mathcal{B}_1$, $E_1 = zz_1\dot{q}_1^2/2$

The Lagrange formalism



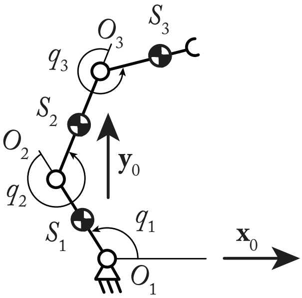
Energy computation:

- For body $\mathcal{B}_1$, $E_1 = zz_1\dot{q}_1^2/2$
- For body $\mathcal{B}_2$, knowing that

$${}^2\mathbf{y}_1 = \begin{bmatrix}\sin q_2 & \cos q_2 & 0\end{bmatrix}^T$$

$$E_2 = \frac{1}{2}\left(m_2\ell_{O_1O_2}^2\dot{q}_1^2 + zz_2(\dot{q}_1+\dot{q}_2)^2\right) + \ell_{O_1O_2}mx_2\dot{q}_1(\dot{q}_1+\dot{q}_2)\cos q_2$$

The Lagrange formalism



Energy computation:

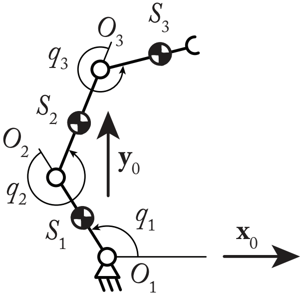
- For body $\mathcal{B}_3$, assuming that $S_3 \equiv O_3$ ($\Rightarrow \mathbf{ms}_3 = \mathbf{0}$), and knowing that

$$\mathbf{v}_3^T \mathbf{v}_3 = \ell_{O_1 O_2}^2 \dot{q}_1^2 + \ell_{O_2 O_3}^2 (\dot{q}_1 + \dot{q}_2)^2 + 2\ell_{O_1 O_2} \ell_{O_2 O_3} \dot{q}_1 (\dot{q}_1 + \dot{q}_2) \cos q_2$$

it comes

$$E_3 = \frac{1}{2} \left(m_3 \ell_{O_1 O_2}^2 \dot{q}_1^2 + zz_3 (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 \right) + m_3 \ell_{O_2 O_3}^2 (\dot{q}_1 + \dot{q}_2)^2 / 2 + m_3 \ell_{O_1 O_2} \ell_{O_2 O_3} \dot{q}_1 (\dot{q}_1 + \dot{q}_2) \cos q_2$$

The Lagrange formalism



Finally

$$
\begin{aligned}
E &= E_1 + E_2 + E_3 \\
&= \frac{1}{2}\left(zz_{1R}\dot{q}_1^2 + zz_{2R}(\dot{q}_1 + \dot{q}_2)^2\right) \\
&+ \frac{1}{2}zz_3(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)^2 \\
&+ lmx_{2R}\dot{q}_1(\dot{q}_1 + \dot{q}_2)\cos q_2
\end{aligned}
$$

where

$$zz_{1R} = zz_1 + m_2\ell_{O_1O_2}^2 + m_3\ell_{O_1O_2}^2$$
$$zz_{2R} = zz_2 + m_3\ell_{O_2O_3}^2$$
$$lmx_{2R} = \ell_{O_1O_2}mx_2 + m_3\ell_{O_1O_2}\ell_{O_2O_3}$$

The Lagrange formalism

Computation of the input torques

Lagrangian $L = E$ (gravity along $\mathbf{z}_0$)

$$\begin{aligned}\tau_1 =& \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} \\ =&(zz_{1R} + zz_{2R} + zz_3)\ddot{q}_1 + (zz_{2R} + zz_3)\ddot{q}_2 + zz_3\ddot{q}_3 \\ &+ lmx_{2R}\left((2\ddot{q}_1 + \ddot{q}_2)\cos q_2 - (2\dot{q}_1 + \dot{q}_2)\dot{q}_2 \sin q_2\right)\end{aligned} \tag{5}$$

$$\begin{aligned}\tau_2 =& \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = (zz_{2R} + zz_3)(\ddot{q}_1 + \ddot{q}_2) \\ &+ zz_3\ddot{q}_3 + lmx_{2R}\left(\ddot{q}_1 \cos q_2 + \dot{q}_1^2 \sin q_2\right)\end{aligned} \tag{6}$$

$$\tau_3 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_3}\right) - \frac{\partial L}{\partial q_3} = zz_3(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3) \tag{7}$$

The Lagrange formalism

This can be put under the following matrix form

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \mathbf{M}\ddot{\mathbf{q}}_t + \mathbf{c} \tag{8}$$

where $\mathbf{q}_t = [q_1 \;\; q_2 \;\; q_3]^T$ and

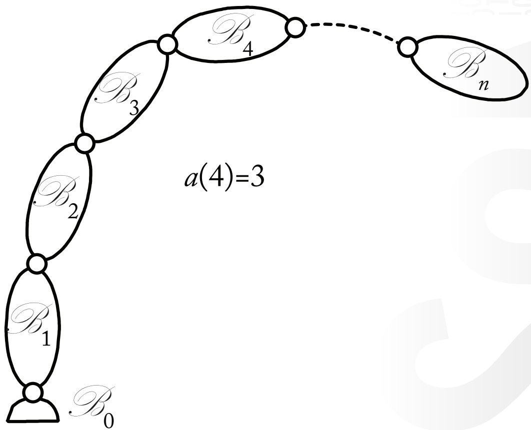
$$\mathbf{M} = \begin{bmatrix} zz_{1R} + zz_{2R} + zz_3 + 2lmx_{2R}\cos q_2 & zz_{2R} + zz_3 + lmx_{2R}\cos q_2 & zz_3 \\ zz_{2R} + zz_3 + lmx_{2R}\cos q_2 & zz_{2R} + zz_3 & zz_3 \\ zz_3 & zz_3 & zz_3 \end{bmatrix} \tag{9}$$

$$\mathbf{c} = \begin{bmatrix} -lmx_{2R}(2\dot{q}_1 + \dot{q}_2)\dot{q}_2 \sin q_2 \\ lmx_{2R}\dot{q}_1^2 \sin q_2 \\ 0 \end{bmatrix} \tag{10}$$

The Lagrange formalism

Generalization

Let us consider a serial robot made of *n* bodies actuated by *n* motors.



The Lagrange formalism

Lagrange equations

$$\boldsymbol{\tau} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_t}\right)^T - \left(\frac{\partial L}{\partial \mathbf{q}_t}\right)^T \tag{11}$$

The Lagrange formalism

Lagrange equations

$$\boldsymbol{\tau} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_t}\right)^T - \left(\frac{\partial L}{\partial \mathbf{q}_t}\right)^T \tag{11}$$

where

- $\mathbf{q}_t$ is the vector of all joint coordinates
- L is the Lagrangian of the robot:

$$L = E - U \tag{12}$$

Computation of the kinetic energy

Kinetic energy of the body $\mathcal{B}_i$

$$E_i = \frac{1}{2} \begin{bmatrix} {}^i\mathbf{v}_i^T & {}^i\boldsymbol{\omega}_i^T \end{bmatrix} \begin{bmatrix} m_i \mathbf{1}_3 & {}^i\widehat{\mathbf{ms}}_i^T \\ {}^i\widehat{\mathbf{ms}}_i & {}^i\mathbf{I}_{O_i} \end{bmatrix} \begin{bmatrix} {}^i\mathbf{v}_i \\ {}^i\boldsymbol{\omega}_i \end{bmatrix} = \frac{1}{2} {}^i\mathbf{t}_i^T \, {}^i\mathbf{M}_i \, {}^i\mathbf{t}_i \qquad (13)$$

Computation of the kinetic energy

Kinetic energy of the body $\mathcal{B}_i$

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Kinetic energy of the whole system

$$E = \sum_{i=1}^{n} E_i = \frac{1}{2} \sum_{i=1}^{n} {}^i\mathbf{t}_i^T \, {}^i\mathbf{M}_i \, {}^i\mathbf{t}_i \qquad (14)$$

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Kinematics of the serial robot

$${}^i\mathbf{t}_i = \mathbf{J}_i \dot{\mathbf{q}}_t \tag{15}$$

Computation of the kinetic energy

Introducing (15) into (14)

$$
\begin{aligned}
E &= \frac{1}{2}\sum_{i=1}^{n}(\mathbf{J}_i\dot{\mathbf{q}}_t)^T\, {}^i\mathbf{M}_i\,(\mathbf{J}_i\dot{\mathbf{q}}_t) \text{ (with } {}^i\mathbf{t}_i = \mathbf{J}_i\dot{\mathbf{q}}_t) \\
&= \frac{1}{2}\dot{\mathbf{q}}_t^T\left(\sum_{i=1}^{n}\mathbf{J}_i^T\, {}^i\mathbf{M}_i\,\mathbf{J}_i\right)\dot{\mathbf{q}}_t \\
&= \frac{1}{2}\dot{\mathbf{q}}_t^T\mathbf{M}_t\dot{\mathbf{q}}_t
\end{aligned}
\tag{16}
$$

with $\mathbf{M}_t = \left(\sum_{i=1}^{n}\mathbf{J}_i^T\, {}^i\mathbf{M}_i\,\mathbf{J}_i\right)$ is the inertia matrix of the robot.

Computation of the kinetic energy

Introducing (15) into (14)

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=& \frac{1}{2} \dot{\mathbf{q}}_t^T \left(\sum_{i=1}^{n} \mathbf{J}_i^T \, {}^i\mathbf{M}_i \, \mathbf{J}_i \right) \dot{\mathbf{q}}_t \\
=& \frac{1}{2} \dot{\mathbf{q}}_t^T \mathbf{M}_t \dot{\mathbf{q}}_t
\end{aligned}
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with $\mathbf{M}_t = \left(\sum_{i=1}^{n} \mathbf{J}_i^T \, {}^i\mathbf{M}_i \, \mathbf{J}_i \right)$ is the inertia matrix of the robot.

Remarks

- $\mathbf{M}_t$ is a $(n \times n)$ symmetric and positive definite matrix

Computation of the kinetic energy

Introducing (15) into (14)

$$\begin{aligned} E =& \frac{1}{2}\sum_{i=1}^{n} (\mathbf{J}_i \dot{\mathbf{q}}_t)^T \, {}^i\mathbf{M}_i \, (\mathbf{J}_i \dot{\mathbf{q}}_t) \text{ (with } {}^i\mathbf{t}_i = \mathbf{J}_i \dot{\mathbf{q}}_t) \\ =& \frac{1}{2}\dot{\mathbf{q}}_t^T \left(\sum_{i=1}^{n} \mathbf{J}_i^T \, {}^i\mathbf{M}_i \, \mathbf{J}_i \right) \dot{\mathbf{q}}_t \\ =& \frac{1}{2}\dot{\mathbf{q}}_t^T \mathbf{M}_t \dot{\mathbf{q}}_t \end{aligned} \tag{16}$$

with $\mathbf{M}_t = \left(\sum_{i=1}^{n} \mathbf{J}_i^T \, {}^i\mathbf{M}_i \, \mathbf{J}_i \right)$ is the inertia matrix of the robot.

Remarks

- $\mathbf{M}_t$ is a $(n \times n)$ symmetric and positive definite matrix
- The kinetic energy of the system is a quadratic function in $\dot{\mathbf{q}}_t$

Computation of the potential energy

Potential energy of the body $\mathcal{B}_i$

$$U_i = -\begin{bmatrix} {}^0\mathbf{g}^T & 0 \end{bmatrix} {}^0\mathbf{T}_i(\mathbf{q}_t) \begin{bmatrix} {}^i\mathbf{ms}_i \\ m_i \end{bmatrix} \quad (17)$$

Computation of the potential energy

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Potential energy of the whole system

$$U = \sum_{i=1}^{n} U_i = -\sum_{i=1}^{n} \left(\begin{bmatrix} {}^0\mathbf{g}^T & 0 \end{bmatrix} {}^0\mathbf{T}_i(\mathbf{q}_t) \begin{bmatrix} {}^i\mathbf{ms}_i \\ m_i \end{bmatrix} \right) \quad (18)$$

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Remarks

Energy expressions (16) and (18):

- are valuable for any types of robots

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Remarks

Energy expressions (16) and (18):

- are valuable for any types of robots
- can be recursively computed

General form of the dynamic equations

Differentiating the Lagrangian, we obtain

$$\boldsymbol{\tau} = \mathbf{M}_t(\mathbf{q}_t)\ddot{\mathbf{q}}_t + \mathbf{C}(\mathbf{q}_t, \dot{\mathbf{q}}_t)\dot{\mathbf{q}}_t + \mathbf{f}_g(\mathbf{q}_t) \tag{19}$$

where:

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where:

- $\boldsymbol{\tau}$ is a $(n \times 1)$ vector of the robot input efforts

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where:

- $\boldsymbol{\tau}$ is a $(n \times 1)$ vector of the robot input efforts
- $\mathbf{C}(\mathbf{q}_t, \dot{\mathbf{q}}_t)\dot{\mathbf{q}}_t$ is the $(n \times 1)$ vector of Coriolis and centrifugal torques, such that:

$$\mathbf{C}(\mathbf{q}_t, \dot{\mathbf{q}}_t)\dot{\mathbf{q}}_t = \dot{\mathbf{M}}_t\dot{\mathbf{q}}_t - \frac{\partial E}{\partial \mathbf{q}_t}$$

General form of the dynamic equations

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- $\mathbf{f}_g(\mathbf{q}_t) = [f_{g1} \dots f_{gn}]^T$ is the vector of gravity generalized efforts.

Systematic comp. of the Coriolis / centrif. torques

The (i, j) element of the matrix $\mathbf{C}$ can be written as:

$$C_{ij} = \sum_{k=1}^{n} c_{i,jk}\dot{q}_k \tag{20}$$

where

$$c_{i,jk} = \frac{1}{2}\left[\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i}\right] \tag{21}$$

and M_{ij} is the element on the ith row and jth column of the matrix $\mathbf{M}_t$

Systematic computation of the vector of gravity generalized efforts

The f_{gi} element of the vector $\mathbf{f}_g$ is calculated according to:

$$f_{gi} = \frac{\partial U}{\partial q_i} \tag{22}$$

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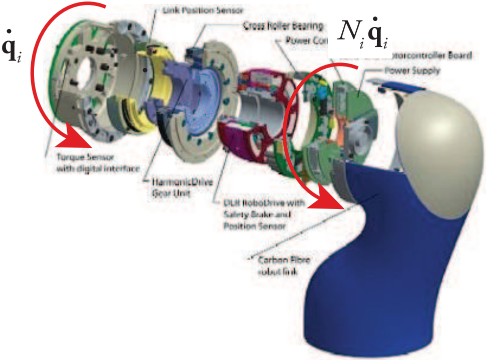
Remark:

The elements of $\mathbf{M}_t$, $\mathbf{C}$ and $\mathbf{f}_g$ are functions of the geometric and inertial parameters of the robot.

Considering the inertia of actuators



N_i is the transmission ratio of joint i axis.

Considering the inertia of actuators

Kinetic energy of the actuator + gearbox

$$E_{act_i} = \frac{1}{2} Ia_i \dot{q}_i^2 \quad (23)$$

where

$$Ia_i = N_i^2 \; Im_i \quad (24)$$

in which

Considering the inertia of actuators

Kinetic energy of the actuator + gearbox

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where

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in which

- Im_i is the moment of inertia of the rotor and transmissions of actuator i,

Considering the inertia of actuators

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in which

- Im_i is the moment of inertia of the rotor and transmissions of actuator i,
- N_i is the transmission ratio of joint i axis.

Remark

In the case of a prismatic joint, Ia_i is an equivalent mass.

Considering the inertia of actuators

General equations

$$\boldsymbol{\tau} = \mathbf{M}_t(\mathbf{q}_t)\ddot{\mathbf{q}}_t + \mathbf{Ia}\,\ddot{\mathbf{q}}_t + \mathbf{C}(\mathbf{q}_t, \dot{\mathbf{q}}_t)\dot{\mathbf{q}}_t + \mathbf{f}_g(\mathbf{q}_t) \tag{25}$$

where $\mathbf{Ia} = diag([Ia_1 \ldots Ia_n])$.

Considering the inertia of actuators

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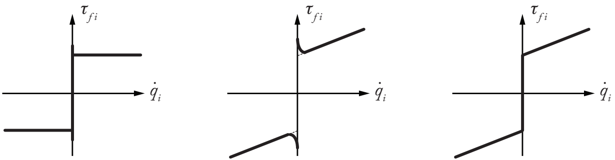
where $\mathbf{Ia} = diag([Ia_1 \ldots Ia_n])$.

Remark

Such modeling neglects the gyroscopic effects of the rotors, which take place when the actuator is fixed on a moving link. However, this approximation is justified for high gear transmission ratios.

Considering friction

Friction models



(a) Static + Coulomb friction model (no dependence of velocity)

(b) Coulomb + viscous friction models (Stribeck effect)

(c) Coulomb + viscous friction models

Figure: Friction models.

Considering friction

Usual model ⇒ (i)

$$\tau_{fi} = fs_i \text{sign}(\dot{q}_i) + fv_i \dot{q}_i \tag{26}$$

Considering friction

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where $\mathbf{F}_v = diag([fv_1 \dots fv_n])$ and $\mathbf{f}_s = [fs_1 \text{sign}(\dot{q}_1) \dots fs_n \text{sign}(\dot{q}_n)]^T$.

Direct dynamic model

The *DDM* that provides the active joint accelerations as a function of the input effort and the active joint positions and velocities is simply described by:

$$\ddot{\mathbf{q}}_t = (\mathbf{M}_t(\mathbf{q}_t) + \mathbf{Ia})^{-1} [\boldsymbol{\tau} - \mathbf{C}(\mathbf{q}_t, \dot{\mathbf{q}}_t)\dot{\mathbf{q}}_t - \mathbf{F}_v\dot{\mathbf{q}}_t - \mathbf{f}_s - \mathbf{f}_g(\mathbf{q}_t)] \quad (28)$$

Concluding remarks

- Lagrange formalism: Easier to use "with the hand"
- Dynamics model can be time consuming $\Rightarrow$ optimize the code computation : Newton-Euler formalism
- Robots with closed-loops, locomotors, etc : a bit different, but another story